QUANTUM COMPUTING

MGS 2025 - Sheffield

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LECTURE 3: Multiple qubit states / gates

The no-cloning theorem

Quantum Teleportation

Multiple Qubit States

The state space for 1 qubit is $\mathbb{C}^2$

The state space for n qubits is: $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} \cong \mathbb{C}^{2^n}$

tensor product

It's the space generated by the tensor products:

$$\left(\alpha_1|0\rangle + \beta_1|1\rangle\right) \otimes \left(\alpha_2|0\rangle + \beta_2|1\rangle\right) \otimes \cdots \otimes \left(\alpha_n|0\rangle + \beta_n|1\rangle\right)$$

If a state can be written in this form, it's called <u>separable</u>

Otherwise it is called <u>entangled</u>

Computational Basis: $|00\cdots0\rangle, \; |0\cdots01\rangle, \; |0\cdots10\rangle, \; \ldots, \; |1\cdots11\rangle$

$$|00\cdots0\rangle = |0\rangle_n, \quad |0\cdots01\rangle = |1\rangle_n, \quad |0\cdots10\rangle = |2\rangle_n, \quad \ldots, \quad |1\cdots11\rangle = |2^n-1\rangle_n$$

decimal number

number of qubits

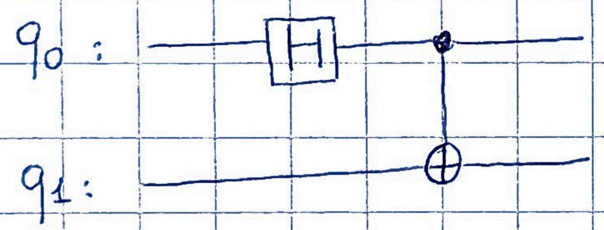
BELL STATES maximally entangled states on 2 qubits:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

$\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ is an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$

Circuit to Create a Bell state:



Exercise: draw circuits to create the other Bell states

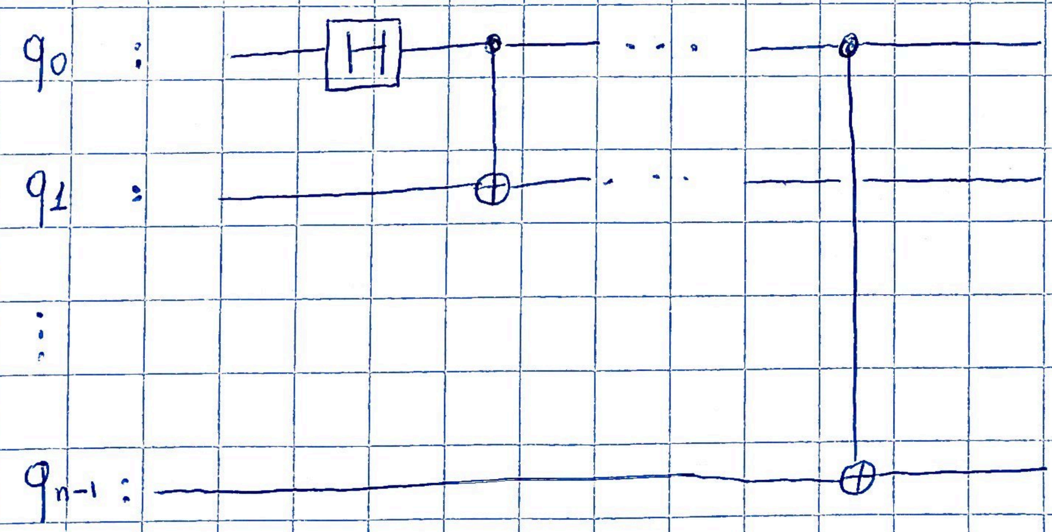
$$|00\rangle \xmapsto{H} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xmapsto{CNOT} \underbrace{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}_{= |\Phi^+\rangle}$$

Generalization to n Qubits:

Greenberger - Horne - Zeilinger (GHZ) states:

$$|GHZ\rangle_n = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\right) = \frac{1}{\sqrt{2}}\left(|0\rangle_n + |2^n-1\rangle_n\right)$$

Created with the GHZ circuit:



THE QUANTUM $H^{\otimes n}$ GATES

Apply the H gate to each qubit in parallel.

For 2 qubits: $H^{\otimes 2}\left(|\Psi_1\rangle \otimes |\Psi_2\rangle\right) = \left(H\,|\Psi_1\rangle\right) \otimes \left(H\,|\Psi_2\rangle\right)$

How to compute tensor products of states and operators:

States:

$$|\Psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle \qquad |\Psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$$

$$|\Psi_1\rangle \otimes |\Psi_2\rangle = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

Operators:

$$A = \begin{bmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{00} & \cdots & b_{0m} \\ \vdots & \ddots & \vdots \\ b_{m0} & \cdots & b_{mm} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{00}B & \cdots & a_{0n}B \\ \vdots & \ddots & \vdots \\ a_{n0}B & \cdots & a_{nn}B \end{bmatrix} = \begin{bmatrix} a_{00}b_{00} & a_{00}b_{0m} & \cdots & \\ a_{00}b_{m0} & a_{00}b_{mm} & & \\ \vdots & & & \\ & & & a_{nn}b_{mm} \end{bmatrix}$$

So, For the Hadamard Gate:

$$H^{\otimes 2} = H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Recursively:

$$H^{\otimes (n+1)} = H \otimes H^{\otimes n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H^{\otimes n} & H^{\otimes n} \\ H^{\otimes n} & -H^{\otimes n} \end{bmatrix}$$

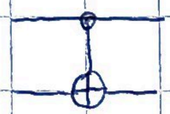
$H^{\otimes n}$ is often used at the beginning of a circuit to prepare the qubits in a state of maximum superposition:

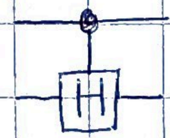
$$H^{\otimes 2}|00\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \qquad H^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}}\left(|0\rangle_n + \cdots + |2^n-1\rangle_n\right)$$

General form of an n-qubit state: $|\psi\rangle = \sum_{j=0}^{2^n-1} \alpha_j |j\rangle_n$ where $\sum_{j=0}^{2^n-1} |a_j|^2 = 1$

So $H^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle_n$

CONTROLLED GATES

CNOT gate  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, CZ gate  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Controlled Hadamard gate: CH  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

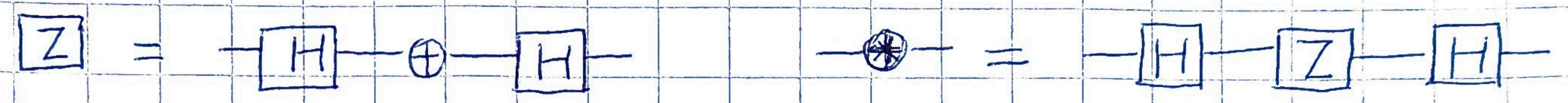
Generalized controlled gate: If gate U has matrix $\begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$

then the controlled U gate has matrix

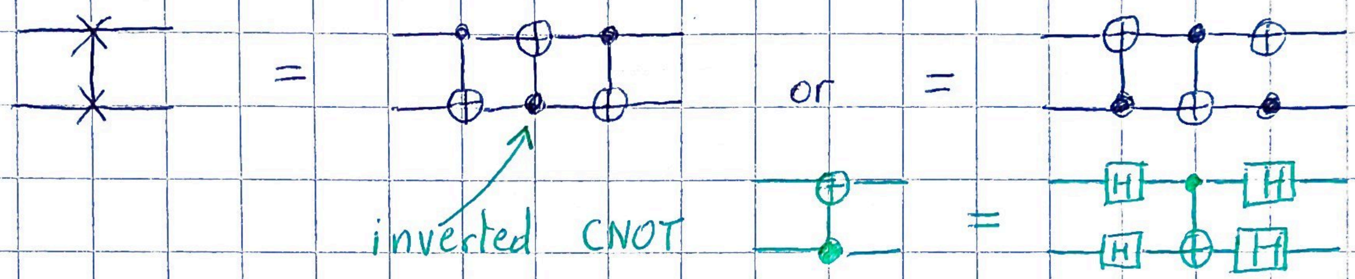
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{11} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

(General way to reason about quantum diagrams: ZX calculus)

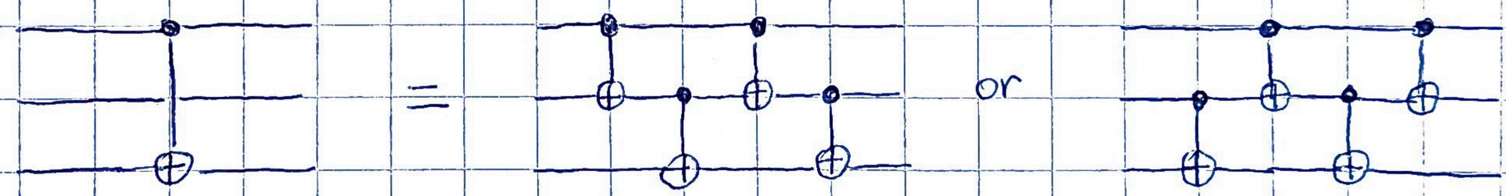
Circuit equivalences (used to simplify circuits)



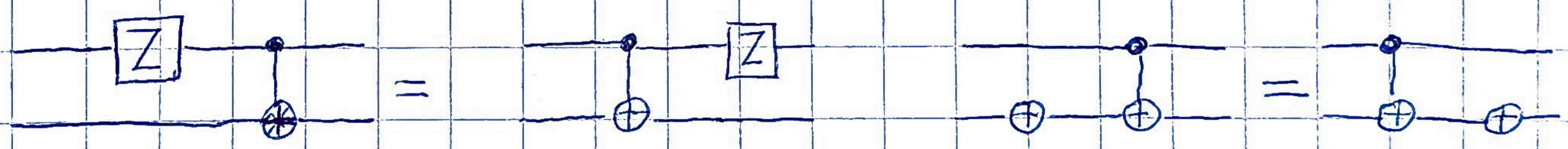
SWAP:



Bridge operations:



Commutation Rules:



NO-CLONING THEOREM

Quantum Information Cannot Be Copied

Assume there is a gate CLONE that can copy the state of a qubit

If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Then $\text{CLONE}\,|\psi 0\rangle = |\psi\psi\rangle$

Left-hand side: $\text{CLONE}\big((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle\big) = \text{CLONE}\big(\alpha|00\rangle + \beta|10\rangle\big)$

$$\underset{\text{linearity}}{=} \alpha \cdot \text{CLONE}\,|00\rangle + \beta \cdot \text{CLONE}\,|10\rangle = \boxed{\alpha|00\rangle + \beta|11\rangle}$$

Righ-hand side: $|\psi\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$

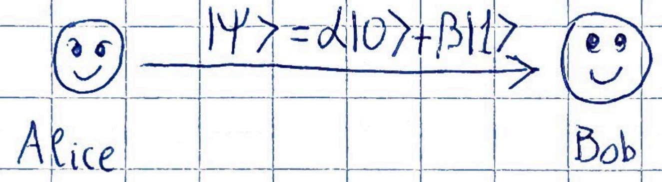
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

the two sides are not generally equal (eg if $\alpha\beta \neq 0$)

QUANTUM TELEPORTATION

Transfer a qubit, preserving state

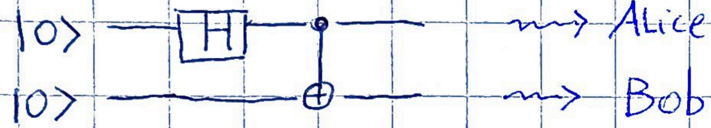
We can transfer a qubit perfectly using: 3 qubits + 2 classical bits



We need:

- 2 entangled bits
  each party has half of the entangled pair
- The message qubit to be teletransported
- A classical communication line
  between the two parties to transmit 2 classical bits
  (so this operation cannot be faster than the speed of light)

<u>STEP 1</u> Create an entangled pair of qubits and distribute it to both parties



$$|00\rangle \longmapsto \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\Phi^+\rangle$$

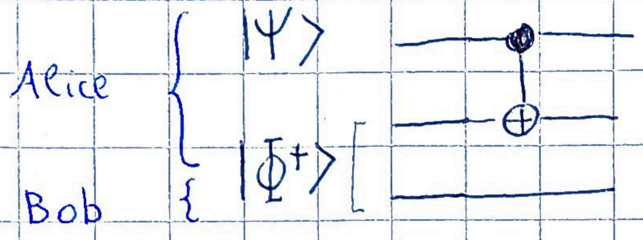
(This is done in advance, maybe before Alice and Bob separate)

Alice also has the message qubit $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Complete quantum state:

$$|\Psi\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)$$

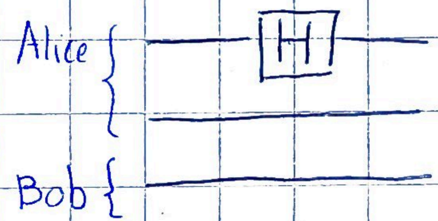
<u>STEP 2</u> Alice applies a CNOT to her qubits:



$$\ldots \mapsto \frac{1}{\sqrt{2}}\left(\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle\right)$$

$$=\frac{1}{\sqrt{2}}\left(\alpha|0\rangle\otimes(|00\rangle+|11\rangle)+\beta|1\rangle\otimes(|10\rangle+|01\rangle)\right)$$

<u>STEP 3</u> Alice applies an Hadamard gate to the message qubit:



$$\ldots \mapsto \frac{1}{\sqrt{2}}\left(\alpha\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes(|00\rangle+|11\rangle)+\beta\frac{1}{2}(|0\rangle-|1\rangle)\otimes(|10\rangle+|01\rangle)\right)$$

$$=\frac{1}{2}\left(\alpha|000\rangle+\alpha|011\rangle+\alpha|100\rangle+\alpha|111\rangle+\right.$$
$$\left.\beta|010\rangle+\beta|001\rangle-\beta|110\rangle-\beta|101\rangle\right)$$

$$=\frac{1}{2}\left(|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle)+|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle)+\right.$$
$$\left.|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle)+|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle)\right)$$

STEP 4 Alice measures her qubits :

There are 4 possible results of the measurement:

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

each with probability $1/4$

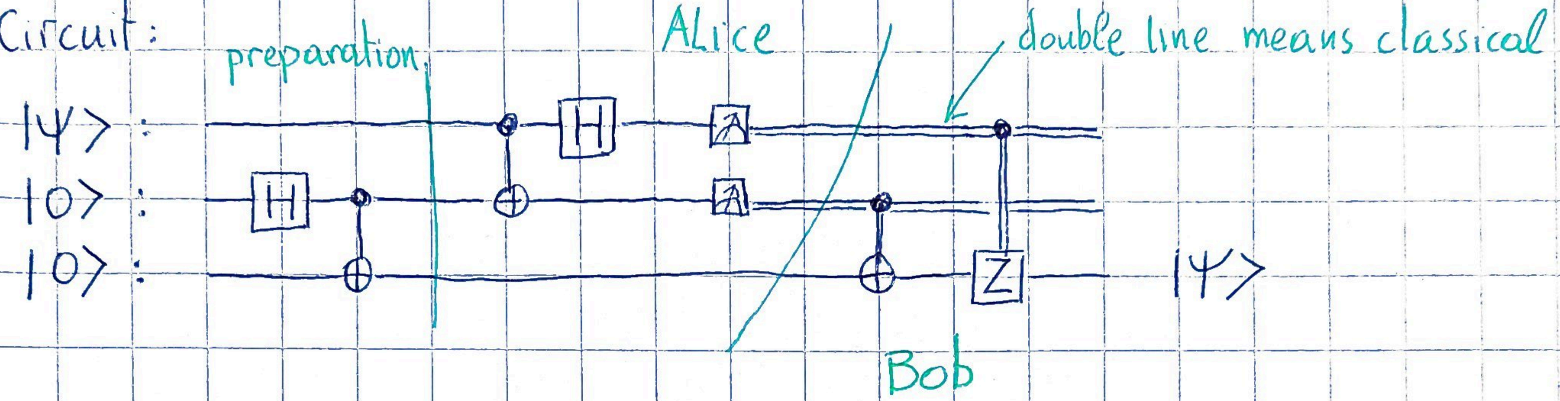
- $|00\rangle \rightsquigarrow$ new state $|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$
- $|01\rangle \rightsquigarrow$ new state $|01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)$
- $|10\rangle \rightsquigarrow$ new state $|10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)$
- $|11\rangle \rightsquigarrow$ new state $|11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)$

STEP 5 Alice transmits the result of the mearurement along the classical channel

STEP 6 Bob applies to his qubit : - a Z gate if the first bit is 1
- a NOT gate if the second bit is 1

This sets Bob's qubit to $|\psi\rangle$

Full Circuit:



Advantages of teleportation:

- We can move a quantum state with just classical communication
- It's easier to move known qubit states (the two entangled qubits) than unknown ones (the message qubit)

Uses:

- Reducing errors
- Linking quantum computers into networks
- Ultra-secure communication channels