QUANTUM COMPUTING

MGS 2025 – Sheffield

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LECTURE 2:

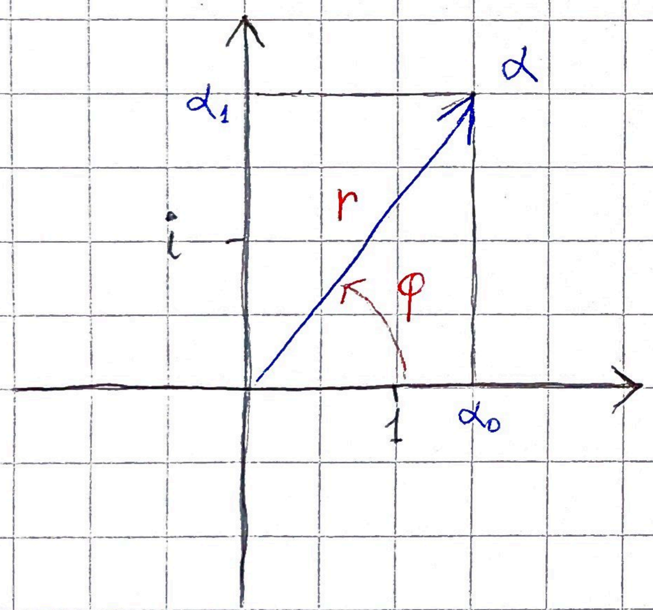
- Some Complex Linear Algebra
- Unitary Matrices – Quantum Operators
- Hermitian Matrices – Observables

A quantum state is represented as a complex linear combination of $|0\rangle$ and $|1\rangle$

$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where α, β are complex numbers

The field of complex numbers is obtained by adding a square root of -1 to the real numbers: $i = \sqrt{-1}$

$\alpha = \alpha_0 + \alpha_1 i$ $\beta = \beta_0 + \beta_1 i$ with $\alpha_0, \alpha_1, \beta_0, \beta_1$ real



Cartesian Coordinates:

$$\alpha = \alpha_0 + \alpha_1 i$$

Polar Coordinates:

$$\alpha = r \cdot e^{\varphi i}$$

So a quantum one-bit state has the form: $|\psi\rangle = r_1 \cdot e^{\varphi_1 i} |0\rangle + r_2 \cdot e^{\varphi_2 i} |1\rangle$

States that differ by a complex unit factor (global phase) are indistinguishable, so we identify them.
Therefore the quantum state can be written with a purely real coefficient for $|0\rangle$:

$$|\psi\rangle = r_1 e^{\varphi_1 i} |0\rangle + r_2 e^{\varphi_2 i} |1\rangle \equiv r_1 |0\rangle + r_2 e^{(\varphi_2 - \varphi_1) i} |1\rangle$$

There are two degrees of freedom:

- r_1 (r_2 is determined by $r_1^2 + r_2^2 = 1$)
- $\varphi = \varphi_2 - \varphi_1$ (relative phase)

We can express r_1 and r_2 as: $r_1 = \cos(\vartheta/2)$, $r_2 = \sin(\vartheta/2)$, $0 \leq \vartheta \leq \pi$

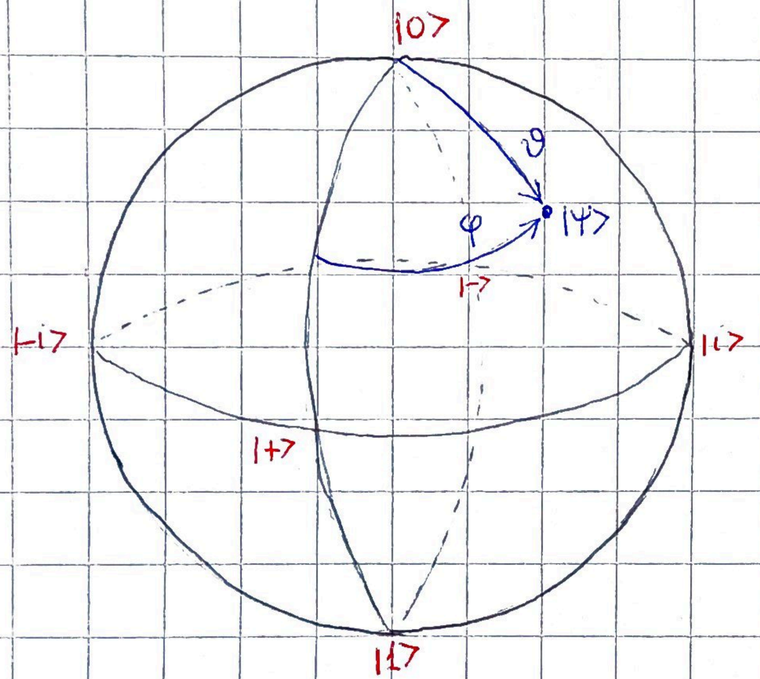
Standard form of a quantum state:

$$|\psi\rangle = \cos\left(\frac{\vartheta}{2}\right) |0\rangle + \sin\left(\frac{\vartheta}{2}\right) e^{\varphi i} |1\rangle$$

The Bloch Sphere

$$|\psi\rangle = \cos\left(\frac{\vartheta}{2}\right)|0\rangle + \sin\left(\frac{\vartheta}{2}\right)e^{\varphi i}|1\rangle \qquad 0 \le \vartheta \le \pi, \quad 0 \le \varphi < 2\pi$$

ϑ and φ determine a point on a sphere:



$|0\rangle$ and $|1\rangle$ are the poles of the sphere:

$|0\rangle : \vartheta = 0, \ \varphi = 0 \qquad |1\rangle : \vartheta = \pi, \ \varphi = 0$

this is called the computational basis

Other antipodal points on the spher correspond to alternative bases

Hadamard Basis: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) : \vartheta = \pi/2, \ \varphi = 0$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) : \vartheta = \pi/2, \ \varphi = \pi$

Circular Basis: $|i\rangle = |\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) : \vartheta = \pi/2, \ \varphi = \pi/2$

$|-i\rangle = |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) : \pi = \pi/2, \ \varphi = 3\pi/2$

<u>QUANTUM OPERATIONS</u> are represented by <u>unitary matrices</u>

- The <u>complex conjugate</u> of a complex number $a+bi$ is
  $\overline{a+bi} = a-bi$

- The <u>conjugate transpose</u> of a matrix $U = \begin{bmatrix} a_{00} & & a_{0m} \\ & \ddots & \\ a_{n0} & & a_{nm} \end{bmatrix}$

  is $U^* = \begin{bmatrix} \overline{a_{00}} & \cdots & \overline{a_{n0}} \\ & & \\ \overline{a_{0n}} & \cdots & \overline{a_{nm}} \end{bmatrix}$

- A matrix is <u>unitary</u> if its conjugate transpose is its inverse:

  square

  $$U^* \cdot U = U \cdot U^* = \mathbb{1} \leftarrow \text{identity matrix}$$

It follows that the columns (and rows) of U are orthonormal:
orthogonal to each orthe and of unit length

So: U is a change of basis (rotation in the Bloch sphere)

The Pauli Gates

- X gate (NOT) $\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a rotation around the X-axis:

  $\sigma_X |0\rangle = |1\rangle$, $\sigma_X |1\rangle = |0\rangle$

  $\sigma_X |+\rangle = |+\rangle$, $\sigma_X |-\rangle = |-\rangle$

  $\sigma_X |i\rangle = |-i\rangle$, $\sigma_X |-i\rangle = |i\rangle$

- Y gate $\sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ is a rotation around the Y-axis:

  $\sigma_Y |0\rangle = |1\rangle$, $\sigma_Y |1\rangle = |0\rangle$

  $\sigma_Y |+\rangle = |-\rangle$, $\sigma_Y |-\rangle = |+\rangle$

  $\sigma_Y |i\rangle = |i\rangle$, $\sigma_Y |-i\rangle = |-i\rangle$

- Z gate $\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a rotation around the Z-axis:

  phase flip

  $\sigma_Z |0\rangle = |0\rangle$, $\sigma_Z |1\rangle = |1\rangle$

  $\sigma_Z |+\rangle = |-\rangle$, $\sigma_Z |-\rangle = |+\rangle$

  $\sigma_Z |i\rangle = |-i\rangle$, $\sigma_Z |-i\rangle = |i\rangle$

Hadamard Gate

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a change of basis between the computational and Hadamard basis

$H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$

$H|+\rangle = |0\rangle$, $H|-\rangle = |1\rangle$

The Hadamard gate is often applied at the begining of an algorithm to prepare the qubits into a superposition.

Phase-Change Gates

Phase-change gates are a generalization of Pauli Z gate that change the relative phase of the qubit by an arbitrary angle φ:

$$R^z_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{\varphi i} \end{bmatrix} \quad \text{for } 0 \leq \varphi < 2\pi$$

$$R^z_0 = \mathbb{1}_2 , \quad R^z_\pi = Z , \quad S = R^z_{\pi/2} , \quad T = R^z_{\pi/4}$$

DENSITY MATRICES

The density matrix of a quantum state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is an alternative representation of the state.

$$\rho = |\Psi\rangle\langle\Psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \times \begin{bmatrix} \bar{\alpha} & \beta \end{bmatrix} = \begin{bmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{bmatrix}$$

If we represent $|\Psi\rangle$ in standard form: $|\Psi\rangle = r_1|0\rangle + r_2 e^{\varphi i}|1\rangle$, $r_1, r_2 \in \mathbb{R}_{\geq 0}$, $0 \leq \varphi < 2\pi$

then $\rho = \begin{bmatrix} r_1^2 & r_1 r_2 e^{-\varphi i} \\ r_1 r_2 e^{\varphi i} & r_2^2 \end{bmatrix}$, so $r_1 = \sqrt{\rho_{00}}$, $r_2 = \sqrt{\rho_{11}}$, $e^{\varphi i} = \rho_{10}/r_1 r_2$

ρ completely determines $|\Psi\rangle$ (up to global phase)

The diagonal elements of ρ are the probabilities of observing $|0\rangle$ and $|1\rangle$

Ex: Density matrix of $|0\rangle$: $|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =: M_0$, of $|1\rangle$: $|1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} =: M_1$

For any $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$\langle\Psi|M_0|\Psi\rangle = |\alpha|^2 \quad , \quad \langle\Psi|M_1|\Psi\rangle = |\beta|^2$$

Measurements

We can set up a measuring device to measure a quantum state along any basis, not just $|0\rangle$ and $|1\rangle$
for example along the Hadamard basis $|+\rangle$, $|-\rangle$

- Choose the value that the measurement should return is a certan state is observed:

  For example: - return 1 if we observe $|+\rangle$
  - return -1 if we observe $|-\rangle$

- The measurement can be fully characterized by a single matrix:
  sum of the density matrices of the observed states with the return values as coefficients:

$$M = 1 \cdot |+\rangle\langle+| - 1 \cdot |-\rangle\langle-| = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is the same as the σ_x operator, but it has a different meaning when used as a measurement

We can recover the values and states from M:
- Observed values = eigenvalues of M
- Observed states = eigenvectors of M

OBSERVABLES: an observable is a matrix which:

- Is Hermitian: $A = A^*$
- The eigenvectors of A form a basis of the quantum state space

An eigenvector is a state that is mapped by A to a multiple of itself

$$A \cdot \underbrace{|\psi\rangle}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} \cdot |\psi\rangle$$

We can compute the eigenvalues by solving the equation

$$\det(A - \lambda \mathbb{1}) = 0$$

and then compute the eigenvectors by substituting the values for λ in.

Several unitary matrices (quantum operations) are also Hermitian (observables)

For example $\sigma_x, \sigma_y, \sigma_z$, H

Exercise 1: Determine what we are observing (eigenvectors) and what the results of the observation are (eigenvalues)

In a real quantum computer we can only observe $|0\rangle$ and $|1\rangle$

~~Show~~

Exercise 2: Show how we can measure any observable by composing with unitary matrices.